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Rapid Poiseuille Flow of Granular Materials

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Summary

We solve the problem of rapid Poiseuille flow of a granular material in a pipe. The material is viewed as an incompressible continuum with a stress given by a Reiner-Rivlin type constitutive equation. Two material parameters appear in the constitutive equation. The magnitude of one of these parameters is related to the flow rate of the granular material in the pipe and the magnitude of an applied pressure gradient.

1. Introduction

Recently Shahinpoor and Lin (1982) and Shahinpoor (1981) have analyzed the rapid couette flow of granular materials. The granular material is viewed as an incompressible continuum with a stress given by a Reiner-Rivlin type constitutive equation. McTigue (1978) had proposed that such a model may be appropriate for analyzing general flows of granular materials. For a general Reiner-Rivlin material (granular material) two material functions appear in the constitutive equation for the stress (see Reiner (1945) and Rivlin (1948)). These functions depend on the principle invariants of the rate of deformation tensor. Shahinpoor and Lin make explicit this dependence so that the stress is compatible with the experimental results of Bagnold (1954). They are able to solve the equations of motion for couette flow between concentric cylinders and by means of a simple experiment they determine one of two material parameters. We investigate here the rapid Poiseuille flow of a granular material in a pipe and illustrate how the magnitude of the remaining undetermined material parameter may be determined.

Poiseuille flows belong to a class of flows which Pipkin and Tanner (1974) call partially controllable viscometric flows. Viscometric flows are ones in which each material element is undergoing a steady simple shearing motion plus possibly a rigid translation and rotation. These viscometric flows can be visualized as the relative sliding motion of a sheaf of inextensible material surfaces (see Lodge (1964), Pipkin (1967) and Yin and Pipkin (1970)). Partially controllable viscometric flows are viscometric flows in which the shape of the slip surfaces is known in advance, but their speeds depend on the form of a viscosity function (see Pipkin and Tanner (1972)). The important feature of these flows is that the normal stress distribution does not influence the velocity distribution. Within this context the couette flow studied by Shahinpoor and Lin (1982) is a partially controllable viscometric flow. The use of a viscometric flow to assess the validity of a given stress response is restrictive since two materials with different stress responses could behave kinematically the same in such a flow. Nevertheless, if we assume that a given stress response is valid then we are at liberty to choose any flow whatsoever as a means of determining any unknown material parameters that can be determined. The validity of our assumptions with regard to stress responses is to be ascertained by comparing theory with experiment for yet more general flows.

2. Equations Governing Rapid Poiseuille Flow of a Granular Material

We imagine a cylinder of radius $r = a$, where generators extend to $z = \pm\infty$, which is filled with a granular material. The granular material is pushed through this cylinder by means of a prescribed pressure gradient at $z \to -\infty$. We assume that the material sticks to the pipe and that the velocity field is steady and rectilinear. The equations describing the motion of this material are given by

$$\varrho\,(\operatorname{grad} u)\,u + \operatorname{grad}\phi = \operatorname{div}\tau, \tag{1}$$

$$\operatorname{div} u = 0, \tag{2}$$

where

$$\phi \equiv P + \varrho g y \tag{3}$$

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is the reduced pressure and P is the hydrodynamic pressure. For incompressible materials P is constitutively indeterminate and is a function only of the temperature. In granular materials it is difficult to define a temperature. However, a pseudo-temperature may be defined which is related to the average kinetic energy of fluctuation of particles. The pressure P in Eq. (3) is assumed to be a function of this pseudo-temperature. Equations (1) and (2) may thus be thought to describe the "average" motion of particles (see Ahmadi and Shahinpoor (1982)). Following Shahinpoor and Lin (1982), we assume the stress in (1) to be given by

$$\tau \equiv T + PI = \alpha_1 |II_D|^{1/2} D + \alpha_2 D^2. \tag{4}$$

Both α_1 and α_2 are material constants. The rate of deformation tensor, D is given by

$$D = \operatorname{grad} u + (\operatorname{grad} u)^T \tag{5}$$

and

$$II_D \equiv \tfrac{1}{2}[(tr\, D)^2 - tr\, D^2] \tag{6}$$

is the second principle invariant of D.

If we particularize relative to a circular cylindrical coordinate system (r, θ, z), then for Poiseuille flow

$$u = w(r)\, e_z, \tag{7}$$

$$D = \tfrac{1}{2} w' (e_r \otimes e_z + e_z \otimes e_r), \tag{8}$$

$$D^2 = \tfrac{1}{4} w'^2 (e_r \otimes e_r + e_z \otimes e_z), \tag{9}$$

$$|II_D|^{1/2} = \tfrac{1}{2}|w'|, \tag{10}$$

$$w' \equiv \frac{dw}{dr}$$

and

$$(\operatorname{grad} u)\, u \equiv 0. \tag{11}$$

The equations of motion in component form become

$$\frac{\partial \phi}{\partial r} = \frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rr}) = \frac{1}{r}\frac{\partial}{\partial r}\left[\frac{\alpha_2}{4} r w'^2\right] \tag{12}$$

and

$$\frac{\partial \phi}{\partial z} = \frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rz}) = \frac{1}{r}\frac{\partial}{\partial r}\left[r \frac{\alpha_1}{4}|w'|w'\right]. \tag{13}$$

The above may be combined to yield

$$\frac{\partial^2 \phi}{\partial r \partial z} = 0 \tag{14}$$

and

$$\frac{\partial}{\partial r}\left\{\frac{1}{r}\frac{\partial}{\partial r}\left[r \frac{\alpha_1}{4}|w'|w'\right]\right\} = 0. \tag{15}$$

The solution to (15) which is bounded at $r = 0$ is

$$|w'|w' = \frac{2c_1}{\alpha_1} r, \tag{16}$$

where c_1 is a constant. The reduced pressure is found from (12) and (14) to be

$$\phi = c_1 z + f(z), \tag{17}$$

where

$$f(r) = \int^r \frac{1}{r}\frac{\partial}{\partial r}\left[\frac{\alpha_1}{4} r w'^2\right] dr. \tag{18}$$

Equation (16) is to be solved relative to the no-slip condition at $r = a$.

From Eq. (16) we see that either $w' \leq 0\ \forall r$ or $w' \geq 0\ \forall r, 0 \leq r \leq a$. The latter case is unphysical for a no-slip boundary condition. Therefore, we take $w' \leq 0\ \forall r, 0 \leq r \leq a$. Equation (16) thus becomes

$$w' = -2^{1/2} \left|\frac{c_1}{\alpha_1}\right|^{1/2} r^{1/2}. \tag{19}$$

The solution to (19) which vanishes at $r = a$ is

$$w = \frac{2}{3}^{3/2} \left|\frac{c_1}{\alpha_1}\right|^{1/2} [a^{3/2} - r^{3/2}] \tag{20}$$

and the reduced pressure is found to be

$$\phi = c_1 z + a_2 \left|\frac{c_1}{\alpha_1}\right| r + D, \qquad D = \text{constant}.$$

If we define the average velocity $\bar{w}$ through

$$\frac{\bar{w} a^2}{2} = \int_0^a r w(r)\, dr,$$

then Eq. (20) becomes

$$w(r) = \frac{7}{3}\bar{w}\left[1 - \left(\frac{r}{a}\right)^{3/2}\right], \tag{21}$$

$$\bar{w} = \frac{2^{3/2}}{7} a^{3/2} \left|\frac{c_1}{\alpha_1}\right|^{1/2}. \tag{22}$$

The volume flux may thus be computed to be

$$Q = \pi a^2 \bar{w} = \frac{\pi}{7} 2^{3/2} a^{7/2} \left|\frac{c_1}{\alpha_1}\right|^{1/2}. \tag{23}$$

From Eq. (17) we identify c_1 to be the applied pressure gradient which may be obtained from measurements of the radial stress at the pipe boundary, $r = a$.

In a completely analogous way to that oulined above we may study the rapid Poiseuille flow of a granular material which is confined between two parallel planes.

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