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On Rapid Flow of Bulk Solids

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Über den schnellen Fluß von Schüttgütern
Ecoulement rapide des solides en vrac
Sobre flujo rápido de sólidos a granel
粉体のラピッドフローについて
论散装固体的快速流动
التدفق السريع للمواد الصلبة السائبة. بقلم ام. شاهيبور.

Summary

A survey is first given on the literature of flow theories of bulk solids as well as the present state of the art on constitutive equations for rapid flow of bulk solids and powders. Then a collision-slip process is defined as the main mechanism responsible in rapid flow theories and a set of pertinent equations are derived for the normal and shear stresses in plane shear rapid flow of bulk solids. Comparing the theoretical results with the available experimental results then reveal the general structure of the stress and the couple stress tensors in rapid flows of bulk solids. An analysis is then presented on the rapid Couette flow of bulk solids and the results of this analysis are compared with some experimental results available in the literature. It appears that rapidly flowing bulk solids do behave similarly to non-Newtonian microfluids and do give rise to normal stress effects as well as non-Newtonian free surface profiles in Couette flows reminiscent of Weissenberg's effects for classical non-Newtonian Couette flows.

Notation

ϱ	bulk density
v_i	velocity vector
$v_{i,j}$	gradient of velocity vector v_i with respect to a fixed spatial coordinate x_j
D_{ij}	deformation rate tensor
τ_{ij}	stress tensor
γ, a	particle density, particle radius
ν	solid volume fraction
μ, μ_s, μ_k	coefficients of friction
p	a fluctuation hydrostatic pressure
δ_{ij}	Kronecker delta
α_1, α_2	material constants
I_2	second deformation rate tensor invariant
P_{yy}, T_{xy}	two dimensional normal and shear stresses
$f(\lambda)$	collision function
λ	linear concentration
α_i	collision angle
u, v, w	particle velocities and microrotation
$\hat{F}_N$	linear momentum
$\hat{F}_f$	friction momentum
m, D	Particle mass and diameter
I	particle moment of inertia
L_d	deformation energy
L_f	friction energy loss
A, B, C	constitutive constants
e	coefficient of restitution
F_c	collision frequency
F_1, F_2, F_3	constitutive parameters
ϵ_{ijk}	permutation symbol
μ_{ij}	couple stress
f_i, c_i	body force and body couple
ω_i	microrotation vector

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1. Introduction

There has recently appeared a great industrial interest in rapid transport of bulk solid materials such as granular materials and powders. Traditionally, bins and hoppers were of wide use in various industrial operations dealing with bulk solids. However, recently, with the advent of utilization of coal as a effective energy source to replace oil there has appeared a great need to understand the rapid transport properties and rheology of powders blown through complex pipe networks and loops. This interest has been contageous and has created a need to understand the rapid flow and transport of powder-like materials in other industrial firms dealing with bulk solids, such as concrete, metal, granular agricultural and pharmaceutical materials, powders, propellant grains, sands, fluidized beds, bulk reactor fuels, solid wastes and many more. Thus, it will become highly desirable to have a set of constitutive equations for powders suitable for their rapid flows in which the main mechanism for transport is continuous particle collisions. Therefore, the purpose of the present research work is, first, to present a mathematical model or a set of constitutive equations for the rapid flow of powders and, second, to analyze the rapid Couette flows of such powders both theoretically and experimentally. Rapid flow refers to those flows which are maintained by continuous particle collisions in an inertia dominated setting with the effects of interparticle as well as boundaries friction and slip taken into consideration.

First, a set of constitutive equations will be derived for the stresses and the couple stesses which predict normal stress effects reminiscent of non-Newtonian fluid behavior. Our preliminary calculations, as presented here, indicate that these constitutive equations resemble in fact non-Newtonian microfluid behaviors in the sense that the stresses depend on the square of both the velocity gradients and particle microrotation gradients.

Furthermore, a tentative calculation on the Couette flow of powder will be presented here in this work to show that the results for the free surface profiles agree with experimental observations and they drastically differ from the classical free surface profiles obtained for both Newtonian and classical non-Newtonian fluids in Couette flows.

In the next two sections we present a survey on the existing literature on the flow of powders as well as the relevant constitutive equations.

2. Literature on Flow Theories

Two idealized flow regimes exist in cohesionless granular materials. The first flow regime which could loosely be termed the slow flow regime or the initial flow regime corresponds to a case in which the interparticle interactions normally arise mainly due to interparticle Coulomb type friction and sliding. In this regime the effect of particle collisions are negligible and one may assume that the constitutive equations are rate independent. Calculations based on this assumption for the steady plastic flows and evolution of slip planes have been carried out by many authors such as Drucker and Parger [1], Shield [2], [3], Drucker, Gibson, and Henkel [4], Jenike and Shield [5], Jenike [6], de Josselin de Jong [7], [8], [9], Shunsuke [10], Rowe [11], Spencer [12], Horne [13], [14], [15], Spencer and Kingston [16], Mandl and Luque [17], Drescher and De Jong [18], Mroz and Drescher [19], Nikolaevskii [20], Drescher [21], Wilde [22], Michalowski and Mroz [23], Mehrabadi and Cowin [24], Nemat-Nasser and Shokouh [25], and Kanatani [26].

In the second flow regime, which is called the rapid flow regime, in contrast to the first slow plastic flow regime, the effect of Coulomb type interparticle friction and interaction is negligible compared to the interparticle forces that arise due to the exchange of momentum in particle collisions. In this regime the nature of constitutive equations are entirely different from the plastic type constitutive equations corresponding to the first regime. Surprisingly very few works have been reported on these types of rapid flows. Historically, Hagen [27] was the first to study the flow of sand in tubes. Bagnold [28], [29] has studied the Couette flow of gravity-free suspension of solid particles in a Newtonian fluid and has defined what is now known as inertia dominated regions in flowing granular materials. Brown and Richards [30] have extensively discussed the various developments on the mechanics and flow of granular materials up to the year 1970. Savage [31] has extensively discussed the literature on various gravity flows of cohesionless granular materials and has mentioned the difficulties encountered in using what is called the Goodman-Cowin continuum theory of granular materials [32], [33], [34] to such flow. Jenkins and Cowin [35] have entensively discussed the various theories for flowing granular materials and have elaborated on the need for acceptable constitutive relationships for the rapid flow of granular materials. Of interest are the works of McTigue [36], Blinowski [37], Ogawa [38], and Kanatani [39], [40], in which new theories for flowing granular materials have been proposed and seem to support Bagnold's observations and results. For further works on the flow theories of granular materials please see Jannan Lee, Cowin, and Templeton III [41], Cowin and Satake [42], Nguyen, Brennen and Sabersky [43], Nunziato, Passman and Thomas Jr. [44], Cowin [45], Passman, Nunziato, Bailey, and Thomas Jr. [46], Shahinpoor and Lin [47], and Shahinpoor and Siah [48], [49].

In the next section I shall present a literature survey on variously proposed constitutive equations for fast flow of granular materials and propose a set of mechanisms to deal with such constitutive equations. In fact we shall derive a new set of constitutive equations for the stress in fast flow of powders and propose ways of obtaining a complete set of constitutive equations for the flow of powders. Having proposed this model, I shall apply a special version of it to the rapid Couette flow of powders between two co-axial rotating cylinders to show that the results obtained for the free surface profile are in agreement with experimental observations, and in fact indicate a kind of non-Newtonian fluid behaviors [50], [51], [52], [53].

3. Literature on Constitutive Equations for Rapid Flow of Powders

The problem of finding suitable constitutive equations for the dependences of forces and couples, in rapidly flowing powder, on the deformation and flow field variables has enjoyed a great attention recently from a number of scientific, engineering, and technological disciplines. Please refer to Brown and Richards [30], Savage [31], Goodman and Cowin [32], [33] Cowin [34], Jenkins and Cowin [35], Cowin and Satake [42], McTigue [36], Blinowski [37], Ogawa [38], Kanatani [39], Shahinpoor and Lin [47], and Shahinpoor and Siah [48], [49], for further information on the rapid flow constitutive theories.

Jenkins and Cowin [35] have extensively discussed the various forms of constitutive equations that may be suitable for the rapid flow of cohesionless granular materials. However, their work is not quite conclusive and in fact they conclude that in a theory for rapidly flowing granular materials the stress should typically have two parts. One part depends explicitly upon the deformation rate tensor D_{ij}, i.e.,

$$D_{ij} = \frac{1}{2} (V_{i,j} + V_{j,i}), \tag{1}$$

where V_i is the velocity vector of the granules and a comma here denotes covariants differentiation with respect to a fixed curvilinear coordinate system X^i. The second part of the stress need not vanish with D_{ij}. Bagnold [28] had concluded that for the inertia dominated regions in rapid simple shear flows of granular materials in absence of gravitational effects the shear and normal stresses are proportional to the square of the rate of deformation D_{ij}. McTigue [36] has employed analytical tools for the classical *billiard ball* problem in treating the shear flow of cohesionless granular materials. His analysis employs a collision frequency function to calculate the net momentum exchange in particle-particle interaction to essentially arrive at results similar to Bagnold's [28]. He concludes that the gravity free parts of stresses are given by:

$$\tau_{12} = \frac{64\gamma a^2}{35\pi} \nu^2 \left| \frac{du_2}{dX_1} \right| \frac{du_2}{dX_1}, \tag{2}$$

$$\tau_{11} = \frac{24\gamma a^2}{35}\,\nu_2\left(\frac{du_2}{dX_1}\right)^2, \tag{3}$$

where X_2 denotes the direction of motion while X_1 is perpendicular to the direction of motion, γ is the particle density, a is its radius and ν is the solid volume fraction.

McTigue [36] has proposed that constitutive equations compatible to the Reiner-Rivlin type constitutive equations (Reiner [50], Rivlin [52]), may be suitable for the general case. However, he does not give clear interpretation of the pressure term that appears in his proposed equation. Kanatani [39] has presented a micropolar continuum theory for the flow of granular materials and in particular the fast flows. He has shown that particle velocity fluctuations play a dominant role in such fast flows and, further, that for inclined gravity flows:

$$\tau_{12} = \frac{3\sqrt{15}}{200}\,C(\varrho)\left(\frac{dv_2}{dx_1}\right)^2, \tag{4}$$

$$\tau_{11} = \frac{\sqrt{6r}}{40\,\mu_a}\,C(\varrho)\left(\frac{dv_2}{dx_1}\right)^2, \tag{5}$$

where r is an occupation radius, $C(\varrho)$ is a given function of density, μ is a kinetic friction coefficient, and a is the particle radius. Blinowski [37] has employed statistical methods developed for the turbulent flow of fluids to describe the granular media rapid flow irregularities and particle fluctuations. From his analysis it is clear that a tensor $K_{ij} \equiv \langle V_i' V_j' \rangle$, i.e., statistical ensemble phase average of the square of particle velocity fluctuations V_i', plays an important role in the constitutive equations. In fact, as shown by Jenkins and Cowin [35], the trace of K_{ij}, i.e., K_{ii} can be shown to represent a pressure like term in the equation for the stress tensor. Ogawa [38], employing a two-temperature theory for granular materials has obtained constitutive equations which depend on the average of the square of the magnitude of the fluctuations in velocity, this being considered as a second temperature in the theory. Although still no clear cut form for the constitutive equations for the rapid flow of granular materials seems to exist, however, it would seem useful to propose one, based on the available experimental and theoretical results, and to check, whether it agrees with the experimentally observed results for a particular rapid flow problem.

In particular, Savage [31], McTigue [36], Jenkins and Cowin [35], Shahinpoor and Lin [47], and Shahinpoor and Siah [48], [49] have discussed the possible form of constitutive equations for the rapid flow of granular materials. Following the line of thoughts of Bagnold [28], McTigue [36], and Jenkins and Cowin [35], Shahinpoor and Lin [47] proposed the following constitutive equation:

$$\tau_{ij} = -p\,\delta_{ij} + \alpha_1 \nu^2 \,|I_2|^{1/2} D_{ij} + \alpha_2 \nu^2 D_{ik} D_{kj}, \tag{6}$$

$$I_2 = \frac{1}{2}\,(D_{mm} D_{nn} - D_{mn} D_{nm}), \tag{7}$$

where p is a dynamic pressure term that is assumed to be due to the statistical trace average of the square of particle velocity fluctuations, i.e., $\langle V_i' V_i' \rangle$ as well as pure hydrostatic effects due to gravitation, α_1 and α_2 are constants, and ν is the solid volume fraction, i.e., $\varrho = \gamma\nu$, where ϱ is the bulk density and γ is the grain density. As can be seen the constitutive equations are compatible with the Reiner-Rivlin type constitutive equations except for the dependence on solid volume fraction ν as well as the role of the ensemble phase trace average $\langle V_i' V_i' \rangle$ in the pressure term p. Equation (6) is also compatible with the experimental results of Bagnold [27] although in his experiment the effect of gravity was absent. We shall elaborate on this point in the next sections and in fact present a fast Couette flow analysis to show that the results predicted from equations (6) and (7) are compatible with experimental results on the Couette flow of powders [47].

McTigue [36], Jenkins amd Cowin [35], Shahinpoor and Lin [47], and Shahinpoor and Siah [48], [49], have also discussed the role of particle collisions in the structure of the constitutive equations for the rapid flow of granular materials in which the main mechanism for the evolution of dispersive forces is random collision or exchange of momentum between pairs of grains. This idea, however, goes back to Bagnold [28], [29] in which he discussed the simple case of a high concentration of large solid spheres in steady shear flows, when the effects of grain inertia dominate and the gravitational and other body forces can be neglected. This is specially true for the case of rapidly flowing granular materials.

Referring to Fig. 1, we consider a simple unidirectional shear motion of a granular aggregate comprising of rigid particles of diameter D, and density γ_0, which are randomly spaced in each layer by parametric distances bD, s, kbD, respectively, and are subject to random pair-wise collisions.

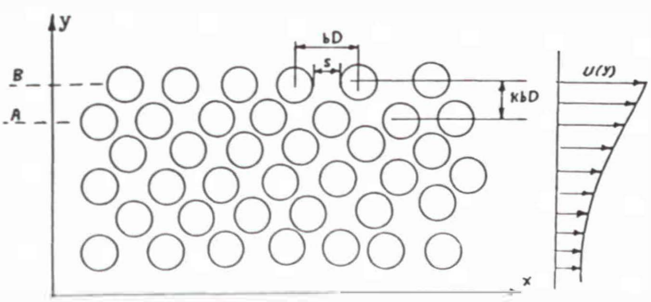

Fig. 1: Cross-section of statistically preferred grain arrangement which might allow dispersive forces resembling normal stress effects in Non-Newtonian fluids

Bagnold [28] calculated that for such flows the dispersive grain forces arising from collisions of pairs of grains, in absence of intergranular friction and gravitational effects, do give rise to a normal stress P_{yy} and a shear stress T_{xy} such that:

$$P_{yy} = a_i\,\gamma_0\,\lambda f(\lambda)\,D^2\left(\frac{du}{dy}\right)^2 \cos\alpha_i, \tag{8}$$

$$T_{xy} = P_{yy}\tan\alpha_i, \tag{9}$$

where $f(\lambda)$ is an unknown function related to the collision frequency, λ is defined as the linear concentration and is given by:

$$\lambda = (D/s), \tag{10}$$

and α_i is the instantaneous angle of collision pertaining to the ith pair of particles (Fig. 2).

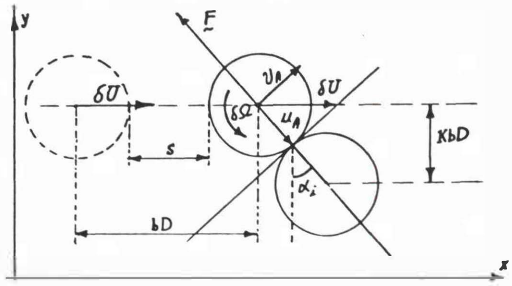

Fig. 2: Geometry of a Random Collision between a pair of spheres as envisaged by Bagnold [28]

Clearly, the angle α_i is a statistical quantity and depends on the collision conditions between pairs of grains. Recently Shahinpoor and Siah [48] have shown that the Bagnold's equations (8) and (9) must actually be multiplied by a sin α_i to correctly describe the situation considered by Bagnold. Thus they obtained:

$$P^*_{yy} = a_i \gamma_0 \lambda f(\lambda) D^2 \left(\frac{du}{dy}\right)^2 \sin\alpha_i \cos\alpha_i \,, \qquad (11)$$

$$T^*_{xy} = P_{yy} \cos\alpha_i = P^*_{yy} \tan\alpha_i \,, \qquad (12)$$

or

$$T^*_{xy} = a_i \gamma_0 \lambda f(\lambda) D^2 \left(\frac{du}{dy}\right)^2 \sin^2\alpha_i \,. \qquad (13)$$

Thus, they concluded that the value of the coefficient a_i, calculated by Bagnold [28] to be equal to 0.042, is actually equal to 0.1378. However, the purpose of the present investigation is to consider both the intergranular contact friction as well as microrotation of grains in pair collisions in fast granular flows to arrive at a set of new constitutive equations for P_{yy} and T_{xy} in simple two dimensional shear flows of granular materials. These constitutive equations relate P_{yy} and T_{xy} not only to the square of velocity gradient $\frac{du}{dy}$ but also to the square of microrotation gradients $\frac{d\omega}{dy}$ as well as the nonlinear product $\left(\frac{du}{dy}\right)\left(\frac{d\omega}{dy}\right)$. In the next section we shall derive these results.

4. The Collision-Slip Process and the Derivation of Governing Equations

In order to arrive at a set of general constitutive equations for a simple plane shear rapid flow of granular materials we consider situations where the grain inertia and pair collisions and slip are the main mechanisms affecting the flow. Referring to Fig. 3 we consider a pair of grains of different diameters, different velocities and different microrotations undergoing what we call a Collision-Slip Process that consists of three stages of *before collision — during collision — after collision*.

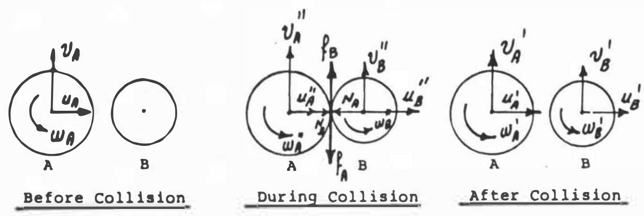

Fig. 3: The collision-slip process for a pair of grains

The Collision-Slip Process allows the grain A to collide with grain B at relative velocities u_A and v_A, and relative microrotation ω_A. We assume that there generally exists some linear and rotational slip between the grains creating a kinematic friction force $F = \mu_k N$ where μ_k is the kinetic friction coefficient. After collision the grains will have new relative velocities $u_A{}'$, $v_A{}'$, $u_B{}'$, $v_B{}'$ and new relative microrotation $\omega_A{}'$ and $\omega_B{}'$. We define a *during collision* interval such that the following relations are approximately correct:

$$u_A{}'' = \frac{1}{2}(u_A + u_A{}'), \qquad v_A{}'' = \frac{1}{2}(v_A + v_A{}'), \qquad (14)$$

$$u_B{}'' = \frac{1}{2} u_B{}', \qquad v_B{}'' = \frac{1}{2} v_B{}' \qquad (15)$$

$$\omega_A{}'' = \frac{1}{2}(\omega_A + \omega_A{}'), \qquad \omega_B{}'' = \frac{1}{2}\omega_B{}'. \qquad (16)$$

The impulsive force (momentum) in the direction of the normal force of collision N is denoted by $\hat{F}_N$ and is given by

$$\hat{F}_N = \int_t N dt = m_A(u_A{}' - u_A) = m_B u_B{}', \qquad (17)$$

which is just a law of conservation of linear momentum in the collision direction.

The impulsive force (momentum) in the direction of the frictional forces f_A and f_B ($f_A = -f_B$) is denoted by $\hat{F}_f$ and is given by

$$\hat{F}_f = \int_t f_A dt = m_A(v_A{}' - v_A) = m_B v_B{}', \qquad (18)$$

where

$$m_A = \frac{\pi}{6}\pi D_A{}^3\gamma_A, \qquad m_B = \frac{\pi}{6} D_B{}^3\gamma_B, \qquad (19)$$

$D_A, D_B \equiv$ grain diameters,

$\gamma_A, \gamma_B \equiv$ grain densities,

and equation (18) is another law for the conservation of linear momentum in the direction perpendicular to collision direction.

The conservation of angular momentum can be written in the following form:

$$\omega_A' - \omega_A = \frac{1}{I_A} \int_t f_A \left(\frac{D_A}{2}\right) dt = (D_A/2I_A)\hat{F}_f, \quad (20)$$

$$\omega_B' = \frac{1}{I_B} \int_t f_B \left(\frac{D_B}{2}\right) dt = (D_B/2I_B)\hat{F}_f, \quad (21)$$

where

$$I_A = \frac{2}{5} m_A (D_A/2)^2 = \frac{1}{10} m_A D_A^2 = (\pi/60) D_A^5 \gamma_A, \quad (22)$$

$$I_B = \frac{2}{5} m_B (D_B/2)^2 = \frac{1}{10} m_B D_B^2 = (\pi/60) D_B^5 \gamma_B. \quad (23)$$

Similarly the conservation of energy equation can be written in the following form:

$$\begin{aligned} & m_A (v_A^2 + u_A^2) + I_A \omega_A^2 = \\ & = m_A (v_A'^2 + u_A'^2) + I_A \omega_A'^2 + m_B (v_B'^2 + u_B'^2) \\ & \quad + I_B \omega_B'^2 + 2L_d + 2L_f, \end{aligned} \quad (24)$$

where L_d is energy loss due to deformation of the grain and is given by

$$L_d = \frac{m_A m_B C^2}{2(m_A + m_B)} (1-e^2), \quad (25)$$

where C is the speed of compression at collision and is given by

$$C = u_A'' - u_B'' = \frac{1}{2}(u_A + u_A' - u_B'), \quad (26)$$

and e is the coefficient of restitution. Further, L_f is the energy loss due to friction and is given by

$$\begin{aligned} L_f &= \int_t \left[\left(v_A'' + \frac{D_A}{2}\omega_A''\right) - \left(v_B'' + \frac{D_B}{2}\omega_B''\right)\right] \mu_k N dt \\ &= \frac{\mu_k}{2} \Big[(v_A' + v_A - v_B') + \\ &\quad + \frac{D_A}{2}(\omega_A' + \omega_A) - \frac{D_B}{2}\omega_B'\Big] \hat{F}_N. \end{aligned} \quad (27)$$

The purpose here is to calculate the six unknowns u_A', u_B', v_A', v_B', ω_A', and ω_B' in terms of the known quantities u_A, v_A, and ω_A which are the relative velocity and micro-rotation components before an impending pair collision between the grains. The six equations for the solutions of the above six unknowns are equations (17), (18), (20), (21), (24), and the last equation is

$$\hat{F}_f = \mu_k \hat{F}_N \quad (28)$$

5. The Solution of Equations

From equations (17) we obtain, respectively, that

$$u_A' = \frac{1}{m_A}\hat{F}_N + u_A \quad (29)$$

$$u_B' = \frac{1}{m_A}\hat{F}_N \quad (30)$$

From equations (18) and (28) we obtain, respectively, that

$$v_A' = \frac{1}{m_A}\hat{F}_f + v_A = \frac{\mu_k}{m_A}\hat{F}_N + v_A, \quad (31)$$

$$v_B' = \frac{1}{m_B}\hat{F}_f = \frac{k}{m_B} F_N \quad (32)$$

From equations (20) and (28), we obtain, respectively, that

$$\omega_A' = (D_A/2I_A)\hat{F}_f + \omega_A = (\mu_k D_A/2I_A)\hat{F}_N + \omega_A, \quad (33)$$

and from equations (21) and (28) we obtain, respectively, that

$$\omega_B' = (D_B/2I_B)\hat{F}_f = (\mu_k D_B/2I_B)\hat{F}_N. \quad (34)$$

From Equations (29), (30), and (26) we obtain

$$C = u_A + \frac{1}{2}\left(\frac{m_B - m_A}{m_A m_B}\right)\hat{F}_N, \quad (35)$$

and from equations (25) and (35) we obtain

$$L_d = \frac{m_A m_B (1-e^2)}{2(m_A + m_B)} \left[u_A + \frac{1}{2}\left(\frac{m_B - m_A}{m_A m_B}\right)\hat{F}_N\right]^2. \quad (36)$$

Furthermore, from equations (27), (31), (32), (33), and 34) we obtain that

$$\begin{aligned} L_f = \mu_k \Big\{ & v_A + \frac{1}{2} D_A \omega_A + \\ & + \frac{1}{2}\mu_k \hat{F}_N \Big[\left(\frac{m_B - m_A}{m_A m_B}\right) + \\ & + \frac{5}{2}\left(\frac{m_B D_B - m_A D_A}{m_A m_B D_A D_B}\right)\Big]\Big\} \hat{F}_N. \end{aligned} \quad (37)$$

Substituting for u_A', u_B', v_A', v_B', ω_A', ω_B', L_d, and L_f from equations (29)—(37) into the energy equation (24) we obtain the following equation for the determination of $\hat{F}_N$ and $\hat{F}_f$.

$$A\hat{F}_N^2 + B\hat{F}_N + C = 0, \quad (38)$$

$$\hat{F}_f = \mu_k \hat{F}_N. \quad (39)$$

where:

$$A = \left(\frac{7}{2}\,\mu_k^2 + 1\right)\left(\frac{m_A + m_B}{m_A\,m_B}\right) + \\ + \frac{1}{2}\,\mu_k^2\left(\frac{m_B - m_A}{m_A\,m_B}\right) + \\ + \frac{5}{4}\,\mu_k^2\left(\frac{m_B D_B - m_A D_A}{m_A\,m_B\,D_A\,D_B}\right) \\ + \frac{1}{8}\left(\frac{m_A^2 + m_B^2 - 2m_A\,m_B}{(m_A + m_B)\,m_A\,m_B}\right)(1-e^2), \tag{40}$$

$$B = 3\,\mu_k\,v_A + 2u_A + \frac{3}{2}\,\mu_k\,D_A\,\omega_A + \\ + \frac{u_A}{2}\left(\frac{m_B - m_A}{m_A + m_B}\right)(1-e^2), \tag{41}$$

$$C = \frac{u_A\,(1-e^2)\,m_A\,m_B}{2\,(m_A + m_B)}. \tag{42}$$

The above equations are too complicated to work with and thus we intend to simplify them for the case of rigid identical grains, i. e., $e=1$, and $m_A = m_B = m = \frac{\pi D^3}{6}\,\gamma_0$. Thus, the dispersive forces $\hat{F}_N$ and $\hat{F}_t$ can be found from equations (31) — (35) to be

$$\hat{F}_N = \left[\frac{-\pi\gamma_0 D^3\,(3\,\mu_k v_A + 2u_A + \frac{3}{2}\,\mu_k D\omega_A)}{(42\,\mu_k^2 + 12)}\right], \tag{43}$$

$$\hat{F}_t = \left[\frac{-\pi\gamma_0 D^3 \mu_k\,(3\,\mu_k v_A + 2u_A + \frac{3}{2}\,\mu_k D\omega_A)}{(42\,\mu_k^2 + 12)}\right]. \tag{44}$$

6. Equations for the Normal and Shear Stresses in Plane Shear Rapid Flow

Let us consider a two-dimensional shear flow field of dry granular materials in the x direction such that both the mean velocity and the mean microrotations of grains are functions of the y coordinate only (Figs. 1 and 3). Thus, the mean relative velocity and the relative microrotations of grains in one layer (layer B, Fig. 1) with respect to the adjacent lower layer (layer A, Fig. 1), can be expressed as (see Fig. 2):

$$\delta U = kbD\left(\frac{du}{dy}\right), \tag{45}$$

such that:

$$\delta U \cos\bar{\alpha}_i = u_A,\ \delta U \sin\bar{\alpha}_i = v_A,\ \bar{\alpha}_i = \frac{\pi}{2} - \alpha_i, \tag{46}$$

$$\delta\omega = kbD\left(\frac{d\omega}{dy}\right) = \omega_A. \tag{47}$$

Thus equations (43) and (44), respectively, reduce to:

$$\hat{F}_N = \frac{-\pi\gamma_0 D^3}{(42\,\mu_k^2 + 12)}\left[(3\,\mu_k \sin\bar{\alpha}_i + \\ + 2\cos\bar{\alpha}_i)(kbD\left(\frac{du}{dy}\right)) + \frac{3}{2}\,\mu_k bD^2\left(\frac{d\omega}{dy}\right)\right], \tag{48}$$

$$\hat{F}_t = \frac{-\pi\mu_k\gamma_0 D^3}{(42\,\mu_k^2 + 12)}\left[(3\,\mu_k \sin\bar{\alpha}_i + \\ + 2\cos\bar{\alpha}_i)(kbD)\left(\frac{du}{dy}\right) + \frac{3}{2}\,\mu_k kbD^2\left(\frac{d\omega}{dy}\right)\right]. \tag{49}$$

Since these forces are associated with one collision alone we must multiply them by the frequency of collision as well as the grain area concentration to arrive at the expressions for the normal and the shear stresses.

In order to calculate the frequency of collision or collision frequency F_c we note that from equations (29), (31) and (43) we have, after some simplification,

$$u_A{}' = \frac{-\left[(3\,\mu_k \sin\bar{\alpha}_i + 2\cos\bar{\alpha}_i)\,\delta U + \frac{3}{2}\,\mu_k D\delta\Omega\right]}{(7\,\mu_k^2 + 2)} + \\ + \delta U \cos\bar{\alpha}_i, \tag{50}$$

$$v_A{}' = \\ = \frac{-\mu_k\left[(3\,\mu_k \sin\bar{\alpha}_i + 2\cos\bar{\alpha}_i)\,\delta U + \frac{3}{2}\,\mu_k^2 D\delta\Omega\right]}{(7\,\mu_k^2 + 2)} + \\ + \delta U \sin\bar{\alpha}_i. \tag{51}$$

Thus, the velocity of grains generally depends on both the δU and $\mu_k D\delta\Omega$. In fact a relative velocity for collision in a mean free path of s can be defined as $\delta U'$ such that

$$\delta U' = u_A{}'/\cos\beta_i = v_A{}'/\sin\beta_i, \tag{52}$$

where β_i is a current angle of collision and is obtained from equations (50), (51) and (52) by means of the following relation:

$$\tan\beta_i = v_A{}'/u_A{}'. \tag{53}$$

It is now clear from equations (50), (51), (52), and (53) that in general

$$\delta U' = A^*\delta U + B^*\,\mu_k D\delta\Omega \tag{54}$$

where A^* and B^* are functions of $\bar{\alpha}_i$ alone. As discussed by Bagnold [28], [29], and Shahinpoor and Siah [48], [49] the collision frequency F_c is proportional to $\delta U'/s$ such that

$$F_c = f(\lambda)\,\frac{\delta U'}{s}, \quad \lambda = \frac{D}{s}, \tag{55}$$

where $f(\lambda)$ is an unknown function of the linear concentration λ. Thus, from equations (45), (46), (47), (52), and (55) we conclude that:

$$F_c = \frac{f(\lambda)}{s}\left[A^* kbD\left(\frac{du}{dy}\right) + B^* kbD^2\left(\frac{d\omega}{dy}\right)\right]. \quad (56)$$

$$T_{xy} = |a_i / b^2 D^2|\, F_c \left[-\hat{F}_N \cos\bar{\alpha}_i - \hat{F}_t \cos\left(\frac{\pi}{2} - \bar{\alpha}_i\right)\right], \quad (57)$$

$$P_{yy} = \left[\frac{\mu_k \cos\bar{\alpha}_i - \sin\bar{\alpha}_i}{\cos\bar{\alpha}_i + \mu_k \sin\bar{\alpha}_i}\right] T_{xy}. \quad (58)$$

Expanding T_{xy}, upon using equations (41), (42), and (49) we obtain

$$T_{xy} = \frac{a_i \pi k^2 \lambda D^2 \gamma_0 f(\lambda)}{(42\mu_k^2 + 12)}(\cos\bar{\alpha}_i + \mu_k \sin\bar{\alpha}_i)\left[F_1(\mu_k, \bar{\alpha}_i)\left(\frac{du}{dy}\right)^2 + F_2(\mu_k, \bar{\alpha}_i) D\left(\frac{du}{dy}\right)\left(\frac{d\omega}{dy}\right) + F_3(\mu_k, \bar{\alpha}_i) D^2\left(\frac{d\omega}{dy}\right)^2\right],$$

where:

$$F_1(\mu_k, \bar{\alpha}_i) = A^*(3\mu_k \sin\bar{\alpha}_i + 2\cos\bar{\alpha}_i), \quad (60)$$

$$F_2(\mu_k, \bar{\alpha}_i) = \frac{3}{2}\mu_k A^* + \mu_k(3\mu_k \sin\bar{\alpha}_i + 2\cos\bar{\alpha}_i) B^*, \quad (61)$$

$$F_3(\mu_k, \bar{\alpha}_i) = \frac{3}{2}\mu_k^2 B^*. \quad (62)$$

The structure of the constitutive equation (52) for T_{xy} is remarkable in the sense that it is a natural generalization of both Bagnold's [28] and McTigue's [36] to include the effect of grain rotation gradient. Moreover, it shows that the shear stress as well as the normal stress in fast flow of granular materials depend on the square of velocity gradient $\left(\frac{du}{dy}\right)$, the square of rotation gradient $\left(\frac{d\omega}{dy}\right)$, as well as the produce $\left(\frac{du}{dy}\right)\left(\frac{d\omega}{dy}\right)$. These equations reduce to the ones obtained by Bagnold [28] if μ_k is set equal to zero. Of course a correction to his equations as discussed by Shahinpoor and Siah [48] should be taken into consideration.

7. Comparison with Experimental Results

For the smooth grain surfaces the friction μ_k can be assumed small and the shear stress should vary proportional to $D^2\left(\frac{du}{dy}\right)^2$ and thus it should vary with D^2 for steady granular flows. This is in agreement with the experimental result of Savage [31], [54]. But a more striking experimental support for equation (59) is the result obtained by Savage [54] for rough wall approximations in which the stresses appear to vary as D^3 rather than D^2. Equation (59) clearly indicates that if friction is appreciable then particle rotations are more predominant and the effect of the coupled term $\left(\frac{du}{dy}\right)\left(\frac{d\omega}{dy}\right)$ becomes more pronounced making the stresses vary as D^3 as is clearly seen from equation (59). Therefore, we conclude that for rougher particles the flow response might heavily depend on the coupled term $\left(\frac{du}{dy}\right)\left(\frac{d\omega}{dy}\right)$ and $\left(\frac{d\omega}{dy}\right)^2$ or the shear stress may vary as D^3 or D^4 or somewhere in between.

8. The Generalized Form of Stresses and Couple Stresses

Since we are considering the microrotation ω as one of the kinematic variables in describing the constitutive equations for the rapid flow of granular materials and powders, we must use the equation of equilibrium, i.e.,

$$\varrho \frac{dv_i}{dt} = \tau_{ij,j} + \varrho f_i, \quad (63)$$

where v_i is the velocity vector, ϱ is the bulk density, f_i is the body force per unit mass, a comma denotes partial differentiation with respect to a fixed rectangular Cartesian system x_i, and τ_{ij} is the generalized stress tensor, as well as the equation of conservation of angular momentum, i.e.,

$$\varrho J \frac{d\omega_i}{dt} = \epsilon_{ijk}\tau_{kj} + \mu_{ji,j} + \varrho C_i, \quad (64)$$

where J is the microinertia, ω_i is the microrotation vector, ϵ_{ijk} is the permutation symbol, C_i is the body couple per unit mass, and μ_{ij} is the couple stress tensor.

Note that so far we have no expression for the couple stress tensor μ_{ij}. Nor do we know if the stress tensor is symmetric or not. These subtle points need to be investigated in the future investigations. My intention is to consider a slightly more general model for particle collisions in order to find some guiding expressions for the couple stress tensor as well as establishing whether or not the generalized stress tensor is symmetric. Once these subtle points are satisfactorily explained then I intend to analyze the rapid Couette and Poiseville flow of powders. In order to show that even with the existing knowledge on the stress tensor, as explained in the present research work, one is able to predict some satisfactory and yet new results on the non-Newtonian behavior of powders in rapid flow, in the next section I present an analysis on the rapid Couette flow of powders.

9. A Tentative Analysis of the Couette Flow of Powders

Using a cylindrical polar coordinate system, we seek a solution to

$$\tau_{ij;j} + \varrho f_i = \varrho \frac{dV_i}{dt}, \qquad ; \equiv \text{Covariant derivative}, \quad (65)$$

where $f_i \equiv (0,0,-g)$, g being the gravitational acceleration, and

$$V_1 = V_r = V_3 = V_z = 0, \quad V_2 = V_\theta = r\omega(r), R_i \leq r \leq R_o. \quad (66)$$

This corresponds to a steady Couette flow between two rotating co-axial cylinders. In order for the considered solutions to be valid the gap between the coaxial cylinder, i.e., R_o—R_i should be sufficiently small to prevent axial, radial and secondary flows. In this case

$$D_{ij} = \frac{1}{2}\begin{bmatrix} 0 & r\omega' & 0 \\ r\omega' & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, D_{ij}D_{jk} = \frac{1}{4}\begin{bmatrix} r^2\omega'^2 & 0 & 0 \\ 0 & r^2\omega'^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad (67)$$

where $\omega' = d\omega/dr$.

Based on the above expressions the components of the stress tensor, according to equation (6), reduce to:

$$\tau_{11} = \tau_{22} = -p + \frac{1}{4}\alpha_2 \nu^2 r^2 \omega'^2 \quad (68)$$

$$\tau_{12} = \frac{1}{4}\alpha_1 \nu^2 |r\omega'| r\omega', \quad (69)$$

$$\tau_{13} = \tau_{23} = 0, \quad \tau_{33} = -p. \quad (70)$$

Note that from the above equations there exists a relationship between the physical components of the normal and shear stress, i.e.,

$$P = \tau_{11} + p = \frac{1}{4}\alpha_2 \nu^2 r^2 \omega'^2, \quad (71)$$

$$T = \tau_{12} = \frac{1}{4}\alpha_1 \nu^2 |r\omega'| r\omega', \quad (72)$$

which are in agreement with both Bagnold's experimental results [28] as well as McTigues's results [36], and Kanatani's results [39]. The governing equations [65] now reduce to:

$$-\varrho r w^2 = \frac{1}{r}\frac{d}{dr}(r\tau_{11}) - \frac{1}{r}\tau_{22}, \quad (73)$$

$$\frac{1}{r^2}\frac{d}{dr}(r^2\tau_{12}) = 0, \quad (74)$$

$$\frac{dp}{dz} = -\varrho g. \quad (75)$$

From equation (75) we obtain:

$$p = -\varrho g z + f(r). \quad (76)$$

Note that we have treated ϱ as a constant which means that in such steady flows the bulk density is assumed to remain constant. As explained by Shahinpoor [55], [56], in this case, the bulk density corresponds to the critical density, i.e., density for loose random packing. This is due to the fact that in such rapid flows all *Voronoi Cells* have equal chance of being created and annihilated, thus given rise to uniform distribution for *Voronoi Cells* or characteristic void spaces.*) This constancy of ϱ or its solid counterpart ν in steady Couette flow is a crucial point in the present analysis.

From equation (74) we obtain that

$$\tau_{12} = A r^{-2}, \quad (77)$$

where A is a constant. From equation (69) and (70) we obtain the following nonlinear differential equation for determining $\omega(r)$:

$$A r^{-2} = \frac{1}{2}\alpha_1 \nu_{cr}^2 |r\omega'| r\omega', \quad (78)$$

where ν_{cr} is the critical solid volume fraction.*) From equations (68) and (73) we obtain the following differential equation

$$-pr\omega^2 = -\frac{dp}{dr} + \frac{1}{4}\alpha_2 \nu_{cr}^2 \frac{d}{dr}(r^2\omega'^2). \quad (79)$$

From equation (76) we obtain that

$$\frac{dp}{dr} = f', \quad \frac{dp}{dr} = -\varrho g \frac{dz}{dr} + f' \quad (80)\,1,2$$

From equations (79) and (80), we obtain

$$f' = \varrho r\omega^2 + \frac{1}{4}\alpha_2 \nu_{cr}^2 \frac{d}{dr}(r^2\omega'^2). \quad (81)$$

Thus,

$$f(r) = \varrho \int r\omega^2 dr + \frac{1}{4}\alpha_2 \nu_{cr}^2 r^2 \omega'^2 + B, \quad (82)$$

where B is another constant of integration. We are now in a position to analyse two distinct cases. First, the case corresponding to the inner cylinder fixed — outer cylinder rotating and second, the case corresponding to the inner cylinder rotating — outer cylinder fixed. In both cases we assume no slip conditions for the velocity V_θ on the boundary walls. This can be achieved in experiments if the contacting walls are made from rubber-like materials or if they are covered with a thin layer of a transparent glue.

* For random aggregates of equal spheres the critical void ratio $e = (1-\nu)/\nu$ equals $e_{cr} \approx 0.64$ under 20 psi overburden pressure (Shahinpoor [55]).

Case 1 — Inner Cylinder Fixed — Outer Cylinder Rotating at a Constant Speed ω_o

In this case $\omega' > 0$ and thus from equation (78) we find that

$$\omega(r) = -2\sqrt{A/\alpha_1}\,\nu_{cr}^{-1} r^{-1} + C, \tag{83}$$

where C is another constant of integration. Surprisingly, the above solution is different from the one obtained for the Couette flow of non-Newtonian fluids by Serrin [53]. Applying the no-slip boundary conditions

$$\omega(R_i) = 0, \quad \omega(R_o) = \omega_o, \tag{84}$$

where R_i and R_o correspond to the inner and outer radii, we obtain from equation (83) that

$$C = 2\sqrt{A/\alpha_1}\,\nu_{cr}^{-1} R_i^{-1};$$
$$A = \frac{\alpha_1 \omega_o{}^2 \nu_{cr}^2}{4}\left(\frac{1}{R_i} - \frac{1}{R_o}\right)^{-2}. \tag{85}$$

Thus,

$$\omega(r) = \omega_o\left(\frac{1}{R_i} - \frac{1}{R_o}\right)^{-1}\left(\frac{1}{R_i} - \frac{1}{r}\right). \tag{86}$$

From equations (85) we note that

$$C = \omega_o \nu_{cr}^{-1} R_i^{-1}\left(\frac{1}{R_i} - \frac{1}{R_o}\right)^{-1} > 0, \quad \frac{A}{\alpha_1} > 0. \tag{87}$$

Shape of the Free Surface: Case 1

Considering equation $(80)_2$ and simplifying that for the free surface, for which p = constant, we obtain:

$$\frac{dz}{dr} = \frac{f'}{\varrho_{cr} g}. \tag{88}$$

Substituting for $f'(r)$ from equation (81) in the above equation yields:

$$\frac{dz}{dr} = \left[\frac{\varrho'_{cr}\omega^2 + \frac{1}{4}\alpha_2 \nu_{cr}^2 \frac{d}{dr}(r^2\omega'^2)}{\varrho_{cr} g}\right]. \tag{89}$$

$$\varrho_{cr} g z = \varrho_{cr}\int r\omega^2 dr + \frac{1}{4}\alpha_2 \nu_{cr}^2 r^2 \omega'^2 + B. \tag{90}$$

This equation can further be simplified to:

$$\rho_{cr} g z = \rho_{cr}\omega_o^2\left(\frac{1}{R_i} - \frac{1}{R_o}\right)^{-2}\left(\frac{r^2}{2R_i^2} + \ln r - \frac{2r}{R_i}\right) + $$
$$+ \frac{1}{4}\alpha_2 \nu_{cr}^2 \omega_o^2\left(\frac{1}{R_i} - \frac{1}{R_o}\right)^{-2} r^{-2} + B. \tag{91}$$

Thus, generally, in this case one winds up with a nonlinear free surface profile which could be tested against experimentally obtained surface profiles for various granular materials to check the validity of the proposed constitutive equations.

Of interest is also the slope of the free surface $\frac{dz}{dr}$ which is given generally, by equation (89) and, specifically, for this case by

$$\frac{dz}{dr} = \varrho_{cr}^{-1} g^{-1}\left[\varrho_{cr}\omega_o^2\left(\frac{r}{R_i^2} + \frac{1}{r} - \frac{2}{R_i}\right)\right.$$
$$\left. - r^{-3}\left(\frac{1}{2}\alpha_2 \nu_{cr}^2 \omega_o^2\right)\right]\left(\frac{1}{R_i} - \frac{1}{R_o}\right)^{-2}. \tag{92}$$

At $r = R_i$ we obtain from equation (92) that:

$$\left.\frac{dz}{dr}\right|_{r = R_i} = $$
$$= -\frac{1}{2}\varrho_{cr}^{-1} g^{-1}\alpha_2 \nu_{cr}^2 \omega_o^2\left(\frac{1}{R_i} - \frac{1}{R_o}\right)^{-2} R_i^{-3}. \tag{93}$$

Thus, depending on the sign of α_2 the fluid tends to climb up or down the inner cylinder, accordingly, if $\alpha_2 > 0$ or $\alpha_2 < 0$, respectively.

Suitable experiments of this kind can be performed to determine the sign of the value of α_2 for any particular granular material. Our experimental result with iodized salt and pulverized coal (Figs. 1 & 2) indicate that $\alpha_2 > 0$, and in fact $\alpha_2 \approx 0.087\ \frac{\text{lb sec}^2}{\text{ft}^2}$.

In fact, one can easily show that generally for the slope of the free surface to be zero at some distance from the center, α_2 must be a positive constant. This can be proved by setting the general expression (89) equal to zero and calculating an r at which $\frac{dz}{dr}$ is equal to zero.

We here present some specific results for this case. For a particle size distribution between 100 to 300 microns, the average diameter was about 189 microns with a standard deviation of about 9%. We randomly created a loose packing of these particles to obtain a $\nu_{cr} \approx 0.5$ according to [55]. The particles weight density was measured to be $\gamma_s \approx$ 119 lb/ft³. For an $\omega_0 = 2\pi$ radians per second, and $R_i = 1''$, $R_0 = 3''$, we measured $\theta_{1(i)} \approx -11°$ and $\theta_{1(0)} \approx 17°$. In order to check the validity of our theory we calculated the value of α_2 from expression (93) and plugged in expression (92) to find the corresponding $\theta_{1(0)}$ to see whether it matches the experimentally measured value of 17°. Thus:

$$\left.\frac{dz}{dr}\right|_{r = R_i} \approx -\tan 11° \approx -0.19438 \approx$$
$$\approx -\frac{1}{2}\gamma_s^{-1}\nu_{cr}\alpha_2\omega_0^2\left(\frac{1}{R_i} - \frac{1}{R_0}\right)^{-2} R_i^{-3}$$

$$\text{or } + 0.19438 \approx$$

$$\approx \frac{0.5 \times 4 \times \pi^2 \times 12}{2 \times 119} \left(1 - \frac{1}{3}\right)^{-2} \alpha_2 \approx 2.2393\, \alpha_2$$

$$\text{or } \alpha_2 \approx 0.0868 \text{ lb sec}^2/\text{ft}.$$

Substituting this value of α_2 in expression (92) we find that:

$$\frac{dz}{dr}\bigg|_{r = R_0} = g^{-1}\omega_0^2 \left[\left(\frac{R_0}{R_i^2} + \frac{1}{R_0} - \frac{2}{R_i}\right) - \right.$$

$$\left. - \frac{1}{2}\varrho_s^{-1}\nu_{cr} R_0^{-3}\alpha_2\right]\left(\frac{1}{R_i} - \frac{1}{R_0}\right)^{-2} \approx 0{,}2993$$

$$\text{or } \tan\theta_{1(0)} \approx 0.2993 \Rightarrow \theta_{1(0)} \approx 16.7^\circ$$

which checks with the experimental results. The location for the minimum z to occur (Fig. 4) is obtained by setting $\frac{dz}{dr}$

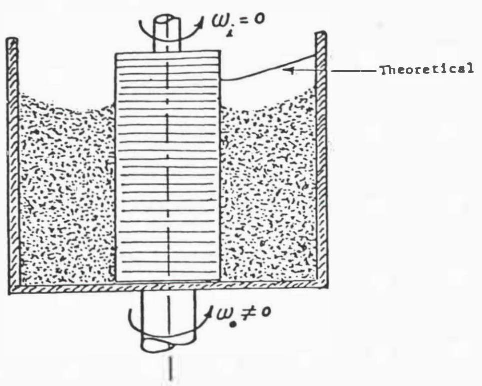

Fig. 4: The shape of the free surface corresponding to case I, i.e., inner cylinder fixed — outer cylinder rotating, $\omega_0 = 2\pi$ rad/sec., $R_i = 1''$, $R_0 = 3''$

equal to zero with α_2 being 0.087 lb sec²/ft² and is found to be at $r \approx 1.0715$ which is different from the experimentally found location as shown in Fig. 4.

We note that the above predictions are entirely different from the ones obtained by Serrin (53) for the Couette flow of a simple non-Newtonian fluid corresponding to this case.

It is interesting to note that the slope at $r = R_0$ depends on the sign of the following expression

$$A^* \equiv \varrho_{cr}\left(\frac{R_0}{R_i} - 1\right)^2 - \frac{1}{2}\alpha_2 \nu_{cr}^2 R_0^{-2}. \qquad (94)$$

Obviously, for sufficiently large R_0/R_i the above expression could be positive as is the case for our experiment (Figs. 4 and 5). However for smaller values of R_0/R_i the above expression could be negative unless:

$$\alpha_2 < 2\,\varrho_{cr}\nu_{cr}^{-2}\left(\frac{R_0}{R_i} - 1\right)^2 R_0^2. \qquad (95)$$

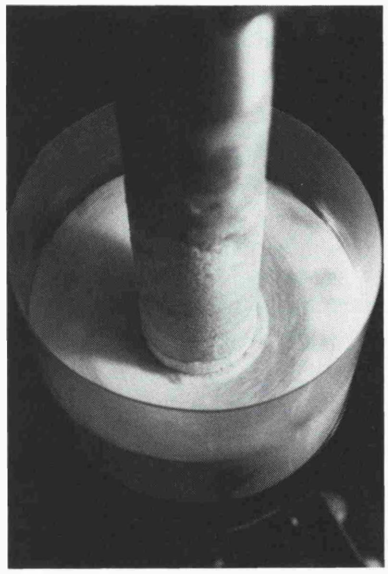

Fig. 5: Photograph of the experimental set up corresponding to Case I

Case 2 —
Inner Cylinder Rotating —
Outer Cylinder Fixed

Since in this case $\omega' < 0$, we find from equation (78) that:

$$A r^{-2} = -\frac{1}{4}\alpha_1 \nu_{cr}^2 r^2 \omega'^2. \qquad (96)$$

Thus, this can be integrated to yield:

$$\omega(r) = -2\sqrt{(-A/\alpha_1)}\,\nu_{cr}^{-1} r^{-1} + C, \qquad (97)$$

where C is a constant of integration. Again this solution is different from the one obtained by Serrin [53] for a simple non-Newtonian fluid corresponding to this special case. Applying the no-slip boundary conditions:

$$\omega(R_i) = \omega_i,\ \omega(R_o) = 0, \qquad (98)$$

we obtain from equation (97) that:

$$C = 2\sqrt{-A/\alpha_1}\,\nu_{cr}^{-1} R_o^{-1},$$

$$A = \frac{-\alpha_1\omega_i^2\nu_{cr}^2}{4}\left(\frac{1}{R_o} - \frac{1}{R_i}\right)^{-2}. \qquad (99)_{1,2}$$

Thus, in this case:

$$\omega(r) = \omega_i\left(\frac{1}{R_o} - \frac{1}{R_i}\right)^{-1}\left(\frac{1}{R_o} - \frac{1}{r}\right). \qquad (100)$$

From equations $(99)_{1,2}$ we note that

$$C = \omega_o \nu_{cr}^{-1} R_o^{-1}\left(\frac{1}{R_o} - \frac{1}{R_i}\right)^{-2} < 0,\ \left(\frac{-A}{\alpha_1}\right) > 0. \quad (101)$$

Shape of the Free Surface: Case II

Considering equation $(80)_2$ again and simplifying it for the free surface, for which p = constant, we obtain:

$$\frac{dz}{dr} = \varrho_{cr}^{-1} g^{-1} \left[\nu_{cr} r \omega^2 + \frac{1}{4} \alpha_2 \nu_{cr}^2 \frac{d}{dr} (r^2 \omega'^2) \right], \quad (89)$$

From equations (89) and (100) we obtain that:

$$z = \varrho_{cr}^{-1} g^{-1} \left[\varrho_{cr} \omega_i^2 \left(\frac{1}{R_o} - \frac{1}{R_i} \right)^{-2} \left(\frac{r^2}{2R_o^2} + \ln r - \frac{2r}{R_o} \right) + \frac{1}{4} \alpha_2^2 \nu_{cr}^2 \omega_i^2 \left(\frac{1}{R_o} - \frac{1}{R_i} \right)^{-2} r^{-2} + B \right]. \quad (102)$$

Of interest again is the slope of the free surface, i.e., $\frac{dz}{dr}$ which is given exactly by:

$$\frac{dz}{dr} = \varrho_{cr}^{-1} g^{-1} \left[\varrho_{cr} \omega_i^2 \left(\frac{1}{R_o} - \frac{1}{R_i} \right)^{-2} \left(\frac{r}{R_o^2} + \frac{1}{r} - \frac{2}{R_o} \right) - \frac{1}{2} \alpha_2 \nu_{cr}^2 \omega_i^2 \left(\frac{1}{R_o} - \frac{1}{R_i} \right)^{-2} r^{-3} \right]. \quad (103)$$

At $r = R_o$ we obtain from equation (103) that:

$$\left. \frac{dz}{dr} \right|_{r = R_o} = - \frac{1}{2} \alpha_2 \nu_{cr}^2 \omega_i^2 \left(\frac{1}{R_i} - \frac{1}{R_o} \right)^{-2} R_o^{-3} g^{-1} \varrho_{cr}^{-1}, \quad (104)$$

and again depending on the sign of α_2 the fluid tends to climb up or down the outer cylinder wall, accordingly, if $\alpha_2 > 0$ or $\alpha_2 < 0$, respectively. Our experimental results with iodized salt and pulverized coal (Figs. 6, 7) indicate

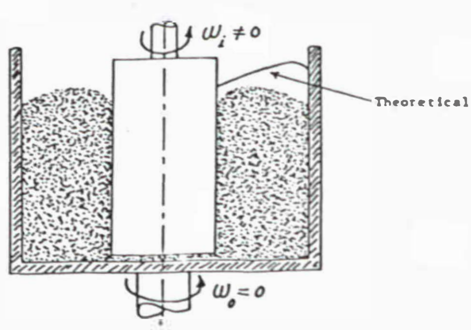

Fig. 6: The shape of the free surface corresponding to Case II, i.e., inner cylinder rotating — outer cylinder fixed, ω_i = 20 π rad/sec., R_i = 1", R_0 = 3"

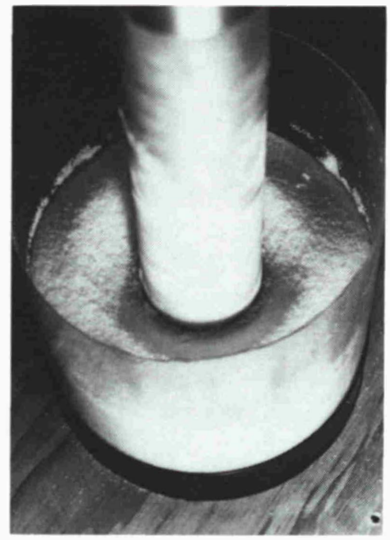

Fig. 7: Photograph of the experimental set up corresponding to Case II

again that $\alpha_2 = 0.087 \frac{\text{lb sec}^2}{\text{ft}^2}$. Again, we here present some specific results concerning this second case. We use the same particles for which $\nu_{cr} \approx 0.5$, $\gamma_s \approx 119$ lb/ft³. For an $\omega_i \approx 20\pi$ radians/second, and R_i = 1", R_0 = 3", we measured $\theta_{2(i)} \approx 43°$, and $\theta_{2(0)} \approx -36°$. In order to check the validity of our theory for this second case we calculated the value of α_2 from expression (104) and substituted it in expression (103) evaluated at $r = R_i$ to find the corresponding $\theta_{2(i)}$ to see whether it matches the experimentally measured value of $\theta_{2(i)}$. Thus

$$\left. \frac{dz}{dr} \right|_{r = R_0} = \tan 36° = -0.72654 =$$

$$= - \frac{1}{2} \gamma_s^{-1} \nu_{cr} \alpha_2 \omega_i^2 \left(\frac{1}{R_i} - \frac{1}{R_0} \right)^{-2} R_0^{-3}$$

$$\text{or } 0.72654 = \frac{0.5 \times 0.5 \times 400 \pi^2 \times 9 \times 12}{4 \times 27 \times 119} \alpha_2 = 8.2938 \alpha_2$$

$$\text{or } \quad \boxed{\alpha_2 \approx 0.0871 \text{ lb sec}^2/\text{ft}^2}$$

which is close to the value found from the experimental results of the first case. From the value of $\theta_{2(i)}$ in expression (103) we find that

$$\left. \frac{dz}{dr} \right|_{r = R_i} = g^{-1} \omega_i^2 \left[\left(\frac{R_i}{R_0^2} + \frac{1}{R_i} - \frac{2}{R_0} \right) - \frac{1}{2} \varrho_s^{-1} \nu_{cr} \alpha_2 R_i^{-3} \right] \left(\frac{1}{R_i} - \frac{1}{R_0} \right)^{-2}$$

$$\approx \tan 43° \approx 0.9325$$

$$\text{or } 10.216982 - 223.932 \alpha_2 = 0.9325 \Rightarrow$$

$$\boxed{\alpha_2 = 0.0415 \frac{\text{lb sec}^2}{\text{ft}^2}}$$

which is not consistent with our previous results. However, the order of magnitude of α_2 is the same as before. Again the location for the maximum z to occur (Fig. 5) is obtained by setting $\frac{dz}{dr}$ equal to zero with α_2 being 0.087 $\frac{\text{lb sec}^2}{\text{ft}^2}$ and is found to be at $r \approx 2.88''$ which is different from the experimentally found location as shown in Fig. 5.

Again, we note that the above predictions are different from the ones obtained by Serrin [53] for the Couette flow of a simple non-Newtonian fluid corresponding to this case. The slope at $r = R_i$ depends on the sign of the following expression:

$$B^* = \varrho_{cr} \left(\frac{R_i}{R_o} - 1 \right)^2 - \frac{1}{2} \alpha_2 \nu_{cr}^2 R_i^{-2} \tag{105}$$

Therefore, for sufficiently small R_i / R_o the above expression could be positive as is the case for our experiment (Figs. 6 and 7).

For larger values of R_i/R_o the above expression (105) can be negative unless:

$$\alpha < 2 \varrho_{cr} \nu_{cr}^{-2} \left(\frac{R_i}{R_o} - 1 \right)^2 R_i^2 \tag{106}$$

Further experimental results are needed for various granular materials to establish the validity of the above conclusions and results.

10. Discussion and Conclusion

The main mechanisms in rapid transport of bulk solids are linear angular momenta transfer by means of which the rapid flow of bulk solids can be explained by a collision-slip process. The effect of interparticulate friction will then naturally be included in a theory of this nature and the final constitutive equations for the stress and the couple stress tensors exhibit non-Newtonian microfluid behaviors. The experimental results obtained on rapid Couette flows of powders indicate good agreement with the results obtained by the theory. Future mathematical modeling of rapid flow of powders should concentrate on finding the microforce and couple fields, i.e., couple stresses μ_{ij}. For the rapid Couette flow experiment the free surface profile must be determined experimentally for various particle size distributions in 140—325 mesh size (40—140 μ), various angular speeds ω_i and ω_o, and various internal and external radii R_i and R_o as explained in the previous section. The determination of the free surface profile in the Couette flow experimental actually presents some challenging problems for the experimenter as we have experienced with our experiments on the rapid Couette flow of iodized salt and coal powders (Figs. 5 and 7). We have tried sonic probes to determine the free surface profile in rapid Couette flow of powders. However, if the bulk solid is electrically conducting such as metallic bulk solids (Alcoa aluminum powders) then simple electrical contact probes seemed to be quite successful in determining the surface profiles of bulk solids. The Poiseuille flow experiment is presently underway and its results will be reported later.

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