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Statistical Mechanical Considerations on Storing Bulk Solids

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Statistisch-Mechanische Betrachtungen zur Speicherung von Schüttgütern

Ein geometrischer Wahrscheinlichkeitsraum für das zu speichernde Schüttgut wird eingeführt und der Begriff der ,Voronoi-Zellen' wird behandelt. Die klassische statistische Theorie und Boltzmann's Postulat werden angewandt auf eine große Ansammlung von Schüttgut ,Voronoi-Zellen'. Exakte Dichte-Wahrscheinlichkeitsfunktionen werden dann für die Verteilung der Porenräume in gespeicherten Schüttgütern aufgestellt. Es wird gezeigt, daß der Anfangszustand eines gespeicherten Schüttgutes einem »willkürlichen, lockeren Speicherzustand« entspricht, bei dem die Porenraumverteilung innerhalb des Schüttgutes gleichmäßig ist. Es wird weiter gezeigt, daß das Konzept der Verdichtung oder Kompaktierung von Schüttgütern durch Vibration oder Lärm einer abnehmenden Entropie und der Verlagerung der Dichte-Verteilung in Richtung auf dichtere Zellen entspricht. Schließlich wird ein Zusammenhang zwischen den »kritischen Zuständen« bei einfacher Scherung des Schüttgutes und dessen Zustand bei willkürlicher, lockerer Speicherung hergestellt.

Considérations mécaniques statistiques du stockage des solides en vrac

On introduit la notion d'un espace géométrique probable pour les solides stockés en vrac et on développe le concept de 'cellules Voronoi' en vrac. On applique la théorie classique et le postulat de Boltzmann à une vaste collection de 'cellules Voronoi' en vrac. On détermine alors des fonctions de probabilité de la densité exactes pour la distribution des vides dans les solides stockés en vrac. On démontre que l'état initial d'un solide entreposé en vrac correspond à un état de «stockage libre» dans lequel la distribution des vides est uniforme dans la masse du solide. On démontre aussi que le concept de tassement des solides stockés en vrac du fait du bruit ou des vibrations correspond à une diminution de l'entropie et au passage de la densité de distribution aux cellules plus denses. Enfin, on établit une relation entre les états critiques atteints lors du cisaillement des solides en vrac et l'ètat de stockage libre.

Consideraciones mecánico-estadísticas del almacenamiento de sólidos a granel

Se introduce un espacio de probabilidad geométrica para materias sólidas almacenadas a granel y se estudia con detalle el concepto de 'celdas Voronoi'. Se aplican la teoría estadística clásica y el postulado de Boltzmann a una colección grande de 'celdas Voronoi'. Se plantean funciones exactas de densidad y probabilidad para la distribución de vacíos en materias sólidas almacenadas a granel. Se muestra que el estado inicial de un sólido almacenado a granel corresponde a un estado de «almacenamiento suelto aleatorio» en el que la distribución de los vacíos es uniforme en todo el producto sólido acumulado a granel. Se muestra asimismo que el concepto de densificación y compactación vibratoria ó por ruido de sólidos almacenados a granel corresponden a una entropía decreciente y al deplazamiento de la densidad de distribución hacia las celdas más densas. Finalmente, se establece una relación entre los estados críticos alcanzados en presencia del esfuerzo cortante sencillo de los sólidos a granel y el estado de almacenamiento en forma aleatoria suelta.

Summary

A geometrical probability space for the stored bulk solids is introduced and the concept of bulk 'Voronoi Cells' is elaborated upon. The classical statistical mechanical theory and the Boltzmann's postulate is specialised to a large collection of bulk 'Voronoi Cells'. Exact probability density functions for the distribution of voids in stored bulk solids are then found. It is shown that the initial state of a stored bulk solid corresponds to a *loose random storing* state for which the void space distribution is uniform throughout the bulk solid. It is further shown that the concept of vibratory and noise densification or compaction of stored bulk solids correspond to a decreasing entropy and the shift of distribution density towards the population of denser cells. Finally, a connection between the critical states reached in simple shearing of bulk solids and the loose random storing-state is established.

Notation

$f(v)$	volume distribution function
v_o	mean particulate volume
$\tilde{\sigma}$	particulate volume standard deviation
e, e_m, e_M, $\bar{e}$	void ratio ≡ ratio of void volume over solid volume
n	porosity ≡ ratio of void volume over total volume
Ω_i	discrete probability
m_i	number of bulk 'Voronoi Cells'
P_i	discrete probability distribution of cells
$p(e)$	probability distribution density
$< >$	ensemble phase average or the expected value
$h_r(e)$	microscopic field functions
S, $\bar{S}$, $\bar{S}_{max}$, $\bar{S}_{cr}$	entropy
$t(e)$	actual distribution density of cells
Z	partition function
λ_r	distribution parameter
$\bar{\varrho}$	bulk solid density
ϱ_s	solid grain density

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1. Introduction

Stored bulk solids form a geometrical probability space for the void space distribution which can be analyzed by statistical mechanical considerations and useful results and conclusions may be obtained on their properties. As explained by Shahinpoor [1] there exists a close link between the random packing and storing of granular materials and bulk solids and the geometrical theory of the structure of fluids.

If the bulk solids are stored randomly there is a great tendency for the void spaces to form a uniform distribution and this essentially corresponds to a state of maximum configuration entropy. This randomly packed space will be unstable in the fields of vibration and shear and tends to densify. The vibratory densification of stored bulk solids forces the uniform void space distribution to become skewed towards the population of smaller void spaces. We shall elaborate on the above concepts in the present paper and introduce the notion of a critical state for a stored bulk solid. The physical correctness of uniform void space distribution density for randomly packed bulk solids may be argued on the basis of the correspondence of the critical state to that of a random loose packing. In this sense a randomly poured body of bulk solids creates equal opportunities, i.e., equal chances of formation, for all possible microarrangements, merely due to its random nature. Mogami [2] has elaborated on some experimental evidence of uniform distribution of void spaces in randomly loose-packed granular assemblies. Finney [3, 4] has discussed this problem in more detail and presented numerical results on pertinent distributions.

In the present paper we intend to present some analytical treatments on the statistical mechanical aspects of bulk solids storage and derive the basic properties of the critical states. However, we must discuss here that there have been traditionally two different approaches to the modeling of bulk materials. One is the *macroscopic* approach which is quite adaptable to various macroscopic experiments such as the measurements of stress, strain, and strain rate. Such treatments were proposed by Drucker and Prager [5], and Spencer [6]. The reader is referred to Brown and Richards [7], and Kezdi [8] for various developments in this regard. The second approach is the *microscopic* or *particulate* approach in which one considers an assembly of rotund particles capable of interacting in the presence of particle collisions and interparticle friction. Mechanical laws concerning the behavior of such aggregates can be deduced by ensemble phase averages of the microscopic character of particle interactions. Reynolds [9] is believed to be the initiator of such an approach for the study of the mechanical properties of bulk solids. Later, Newland and Allely [10], Rowe [11], Horne [12, 13] added new contributions with regard to the particulate approach. Since the mechanical properties of bulk solids are very much dependent on the distribution of void spaces as well as the particles themselves, one must resort to statistical ensemble phase average approaches in order to obtain meaningful results. Magami [2, 14] was the first to consider such statistical approaches for the mechanical behavior of bulk solids. His approach had limited success.

In the next section we shall discuss the geometrical probability distribution space for bulk solids and introduce the notions of the bulk 'Voronoi Cells' and the entropy of spatial configurations. In the process we shall discuss the relevance of Boltzmann's postulate [15] to the basic characteristics of bulk solids.

2. Geometrical Probability Distribution Space for Bulk Solids

Let us consider an aggregate of bulk solids composed ideally of rotund particles with a volume distribution function $f(v)$ such that $f(v)$ can be assumed to be Gaussian, i.e.,

$$f(v) = \frac{1}{\tilde{\sigma}\sqrt{2\pi}} \exp\left[-(v-v_o)^2/2\,\tilde{\sigma}^2\right], \tag{1}$$

where v_o is the average particle volume and $\tilde{\sigma}$ is the volumetric standard deviation. We assume this aggregate of bulk solids is composed of cohesionless particles bound together by some external force systems such as gravity and boundary tractions or simple hydrostatic pressure states. We further assume that the container boundary's geometrical constraints play minimal effects on the stored bulk solids' geometrical distributions. The various types of packing that can be produced by random pouring of bulk solids in large containers have been discussed by Finney [3], based on the previous results of Bernal and coworkers [1]. In the absence of gravitational and frictional effects the range of critical porosities has been found, for equal spheres, to be $0.363 \leqq n_{cr} \leqq 0.391$. Microscopically speaking, the void ratio e which is related to porosity n by

$$e = n/1-n, \tag{2}$$

will not be uniform throughout the stored bulk solids aggregate but rather it will have a geometrical distribution. We adapt the line of approach of Finney [3] in considering the bulk 'Voronoi polyhedron' as the unit all for the aggregate of bulk solids. This cell has the added advantage that there is only one polyhedron associated with each center, i.e., particle. Fig. 1 shows some typical corresponding cells for a two-dimensional aggregate of equal hard disks.

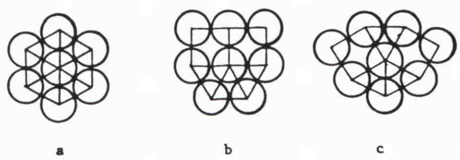


Fig. 1: Typical two-dimensional bulk Voronoi Cells

We employ the bulk 'Voronoi polyhedra' here for our statistical mechanical analysis of stored bulk solids. Each cell is also called a characteristics microelement. The volume of each single bulk 'Voronoi polyhedron' is denoted by V_i and thus the void ratio associated with each unit cell is e_i such that

$$e_i = \frac{V_i - V_s^{(i)}}{V_s^{(i)}} \tag{3}$$

where $V_s^{(i)}$ is the solid volume associated with each bulk 'Voronoi Cell'. Two or more cells with completely different bulk polyhedral geometries may have the same void ratio e_i. If m_i be the number of cells in a unit volume of bulk solids having a void ratio e_i, then we may define a probability distribution Ω_i that

$$\Omega_i = m_i \Big/ \sum_{i=1}^{N} m_i, \tag{4}$$

$$\sum_{i=1}^{N} \Omega_i = 1, \tag{5}$$

where Ω_i is the probability of formation of a group of m_i cells, scattered throughout the unit volume, having a void ratio e_i, and N is the total number of such groups in such a way that e_i's, $i = 1, 2, \ldots, N$, arranged in an ascending order span a void ratio space from $e_1 = e_{min} = e_m$ to $e_N = e_{max} = e_M$. The expected value of the void ratio $\bar{e}$ of the bulk solid aggregate is then obtained as

$$\bar{e} = \sum_{i=1}^{N} \Omega_i \, e_i \,. \tag{6}$$

If the number of groups of bulk 'Voronoi Cells' is large enough then one may assume that as $N \to \infty$ one may define a continuous probability density $p(e)$ such that

$$P_i = p(e_i)\,(e_{i+1} - e_i) = p(e_i)\,\Delta e_i \,, \tag{7}$$

or simply

$$P = p(e)\, de \,, \tag{8}$$

$$\int_{e_m}^{e_M} p(e)\, de = 1 \,, \tag{9}$$

$$\bar{e} = \int_{e_m}^{e_M} p(e)\, e\, de = \langle e \rangle \,, \tag{10}$$

where $\langle e \rangle$ denotes the expected value of bulk solids void ratio $\bar{e}$. Let us note that the limits on the void ratio e, i.e., $e_m \leqq e \leqq e_M$ depend on a number of factors. These include the gravitational as well as interparticle frictional effects. The particulate geometry also plays a role. In the absence of gravitational and frictional effect, e_m would correspond to the densest geometrical arrangements of the cells. In the case of hard spherical particles, e_m is a known quantity ($e_m \approx 0.3504$) corresponding to a coordination number 12, i.e., that of the densest rhombohedral packing. It is well known, that e_M depends very much on the existing confining or overburden pressure on the bulk solids; such pressures produce internal shear failures on slip planes and render loose arrangements unstable. Therefore, e_M should be very sensitive to interparticle friction angles. Haruyama [16] has presented some clear results on the effect of particulate surface roughness on the packing of bulk solids. The more the interparticle friction the larger the values of e_m and e_M will become in typical packings of bulk solids. In the absence of frictional effects, for hard spherical particles e_M is roughly $e_M \approx 0.90985$ corresponding to the loose cubic packing having a coordination number 6. Fig. 2 shows the geometrical arrangements of spheres corresponding to the above two limits. Tables 1 and 2 display, respectively, the typical three-dimensional and two-dimensional bulk 'Voronoi Cells'. For aggregates of bulk solids with a size distribution the number of possible 'Voronoi Cells' becomes extremely large and the use of continuous distribution becomes more justified.

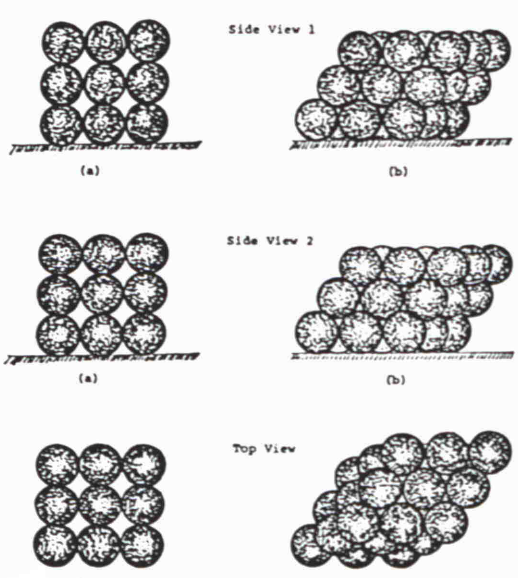


Fig. 2: Cubic (a) and rhombohedral (b) packing of spheres

Coordination Number	Symbol	Porosity	Void Ratio	Cell Number
4	[1,1,2]	0.7181	2.5473	1
	[1,0,3]	0.6599	1.9403	2
	[2,0,2]	0.6599	1.9403	3
5	[1,2,2]	0.6933	2.2605	4
	[2,1,2]	0.6298	1.7012	5
	[1,2,2]≡[1,3,1]	0.5969	1.4808	6
	[1,1,3]	0.5790	1.3753	7
	[1,0,4]	0.5582	1.2635	8
	[2,0,3]	0.5373	1.1612	9
6	[2,2,2]	0.6298	1.7017	10
	[2,2,2]	0.5578	1.2614	11
	[1,2,3]	0.5574	1.2594	12
	[1,3,2]	0.5546	1.2452	13
	[2,2,2]≡[3,0,3]≡[1,4,1]	0.4764	0.9098	14
	[2,0,4]	0.4746	0.9033	15
7	[1,2,4]≡[2,2,3]	0.5132	1.0542	16
	[2,3,2]	0.5063	1.0255	17
	[3,1,3]	0.4918	0.9677	18
	[1,4,2]≡[1,5,1]	0.4389	0.7822	19
8	[2,2,4]	0.4642	0.8664	20
	[1,5,2]	0.3985	0.6625	21
	[2,2,4]	0.3954	0.6540	22
	[3,2,3]≡[2,4,2]≡[1,6,1]	0.3954	0.6529	23
	[4,0,4]	0.3198	0.4702	24
9	[1,4,4]	0.3866	0.6303	25
	[2,5,2]≡[1,6,2]	0.3520	0.5432	26
10	[2,4,4]≡[1,6,3]	0.3343	0.5022	27
	[2,5,3]	0.3127	0.4550	28
	[4,2,4]≡[2,6,2]	0.3019	0.4327	29
11	[2,6,3]	0.2813	0.3908	30
12	[3,6,3]≡[4,4,4]	0.2595	0.3504	31

Table 1: Some typical three-dimensional bulk 'Voronoi Cells'

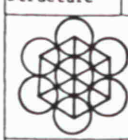
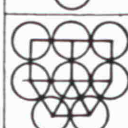
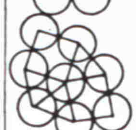
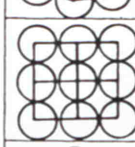
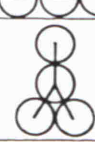
Cellular Structure	Coordination Number	Porosity	Void Ratio	Cell Number
	6	0.0931	0.1027	1
	5	0.1582	0.1879	2
	5	0.1582	0.1879	3
	4	0.2146	0.2732	4
	3	0.3954	0.6540	5

Table 2: Some typical two-dimensional bulk 'Voronoi Cells'

3. Statistical Mechanical Procedure Applied to Stored Bulk Solids

In order to find the gross mechanical behavior of bulk solids from microscopic properties it is necessary to define an ensemble phase average $\bar{Q}$ denoted by $\langle Q\rangle$. Here Q is any microscopic measurable quantity $Q(e)$ which is a function of the void ratio e, such that

$$\bar{Q} \equiv \langle Q\rangle \equiv \int_{e_m}^{e_M} p(e)\, Q(e)\, de. \tag{11}$$

We note that $Q(e)$ can also be interpreted as the value of a particular measurable property for a common group of bulk 'Voronoi Cells'. It is straight forward to generalize the above definition to include other microscopic variables such as the particle size and volume, coordination number, or contact angles. Essentially we must have a knowledge of pertinent distribution densities. It is interesting to note that, essentially for most practical applications, the above distributions, except for $p(e)$, are Gaussian. Thus for any microscopic variable $G(e, v, \theta, N)$, where v is particulate volume, θ is the particulate contact angle, and N is the coordination number we may define a gross average $\bar{\bar{G}} \equiv \langle\langle\langle\langle G \rangle\rangle\rangle\rangle$ such that

$$\bar{\bar{G}} \equiv \int_{e_m}^{e_M} \int_{v_m}^{v_M} \int_{0}^{2\pi} \int_{N_m}^{N_M} p(e)\, f(v)\, \Theta(\theta)\, g(N)\, G de dv d\theta dN, \tag{12}$$

where $\Theta(\theta)$ and $g(N)$ are, respectively, the distribution densities on contact angles and coordination number, v_m and v_M are, respectively, the minimum and maximum particulate volume, and N_m, N_M are, respectively, the minimum, maximum coordination numbers. If R macroscopic average $\langle h_r(e)\rangle$, $r = 1, 2, \ldots, R$, of microscopic field functions $h_r(e)$ associated with groups of bulk 'Voronoi Cells' can be measured for an aggregate of bulk solids, then the equations:

$$\langle h_r(e)\rangle = \int_{e_m}^{e_M} p(e)\, h_r(e)\, de, \; r = 1, 2, \ldots, R, \tag{13}$$

would serve as side constraints on any macroscopic extremization problem associated with such a medium.

According to the Boltzmann's postulate [15, 17] one may link the microscopic statistical theory and the macroscopic thermodynamics by considering the entropy $\bar{S}$ such that $\bar{S}$ is in fact the expected value $\langle S\rangle$ of the microscopic entropy S which is proportional to the logarithm of the total number of possible or probable configurations that can be obtained by arranging N groups of bulk 'Voronoi Cells' in a given volume. In our case

$$\bar{S} = -k\, \langle \ell n\, p\, (e)\rangle, k < 0, \tag{14}$$

or

$$\bar{S} = -k \int_{e_m}^{e_M} p(e)\, \ell n\, p(e)\, de. \tag{15}$$

It has been argued by Jaynes [18, 19] and Kanatani [20] that the correct representation of entropy should be

$$\bar{H} = -\int_{e_m}^{e_M} p(e)\, \ell n\, (p(e)/t(e))\, de, \tag{16}$$

where $t(e)$ is the actual distribution density of states as compared to the probability distribution density $p(e)$ of states. Since obviously expression (16) represents a certain distance between the two distributions thus it may be considered as a measure of prejudice or bias of one distribution against the other. There appears to exist some confusion in the pertinent literature on this concept and on whether, $\bar{H}$, the Shannon's entropy [21] is related to the statistical mechanical entropy $\bar{S}$. We shall choose not to go into such discussions in this paper and instead treat $\bar{S}$ as our pertinent macroscopic entropy.

The classical Maxwell-Boltzmann equation for discrete systems possesses solutions which describe the irreversible movement of the aggregate towards equilibrium in such a manner that

$$\frac{\partial \bar{S}}{\partial t} > 0 \tag{17}$$

so that finally at equilibrium

$$\frac{\partial \bar{S}}{\partial t} = 0, \text{(at equilibrium)}. \tag{18}$$

So that the entropy $\bar{S}$ is a maximum at any equilibrium state. This is, of course, in agreement with the second law of thermodynamics. We assume that the above arguments apply to stored aggregates of bulk solids in the sense that any statistically equilibrated random aggregate of bulk solids of average void ratio $\bar{e}$ is an equilibrated state possessing a maximum possible average entropy $\bar{S}$ corresponding to $\bar{e}$. Given a set of expectation values $\bar{h}_r \equiv \langle h_r(e)\rangle$, $r = 1, 2, \ldots, R$ for microscopic measurable quantities $h_r(e)$ such that

$$\bar{h}_r = \langle h_r(e)\rangle = \int_{e_m}^{e_M} p(e)\, h_r(e)\, de, \; r = 1, 2, \ldots, R, \tag{19}$$

$$\int_{e_m}^{e_M} p(e)\, de = 1, \tag{20}$$

the equilibrium distribution density $p(e)$, for the bulk solid aggregate, which maximizes $\bar{S}$ can be shown to be given by

$$p(e) = \frac{1}{Z} \exp\left[-\sum_{r=1}^{R} \lambda_r h_r(e)\right], \tag{21}$$

where the partition function Z is given by the following equation

$$Z(\lambda_r) = \int_{e_m}^{e_M} \exp\left[-\sum_{r=1}^{R} \lambda_r h_r(e)\right] de, \tag{22}$$

and the λ_r's, $r = 1, 2, \ldots, R$, are the solutions to the following transcendental equations

$$\bar{h}_r = \frac{\partial}{\partial \lambda_r} [\ell n Z(\lambda_r)], \; r = 1, 2, \ldots, R. \tag{23}$$

We further note that

$$\bar{S}_{max} = k \sum_{r=1}^{R} \left[-\lambda_r \frac{\partial}{\partial \lambda_r} [\ell n Z(\lambda_r)] + \ell n Z(\lambda_r)\right]. \tag{24}$$

If $\bar{e} \equiv \langle e \rangle$ is the only gross quantity that can be measured, then

$$p(e) = \lambda_1 \exp[-\lambda_1 e]/(\exp[-\lambda_1 e_m] - \exp[-\lambda_1 e_M]), \quad (25)$$

such that λ_1 can be found from the following transcendental equation:

$$\bar{e} = \lambda_1^{-1} + \left\{ \frac{e_m \exp[-\lambda_1 e_m] - e_M \exp[-\lambda_1 e_M]}{\exp[-\lambda_1 e_m] - \exp[-\lambda_1 e_M]} \right\}. \quad (26)$$

From equation (24) we observe that

$$\frac{\partial \bar{S}_{max}}{\partial \lambda_r} = -k \sum_{r=1}^{R} \lambda_r \frac{\partial_2}{\partial \lambda_r^2} [\ell n Z(\lambda_r)], \quad (27)$$

and thus, among all distributions, that tend to maximize $\bar{S}$, the uniform distribution tends to maximize $\bar{S}$ the most. For this distribution

$$\frac{\partial \bar{S}_{max}}{\partial \lambda_r} = 0 \Rightarrow \lambda_r = 0 \forall r = 1, 2, \ldots, R. \quad (28)$$

We call the value of $\bar{S}_{max}$ corresponding to such a uniform distribution the critical value and note that

$$\bar{S}_{cr} = \sup \bar{S} = \max \bar{S}_{max} = k \ell n (e_M - e_m), \quad (29)$$

$$p_{cr} = p_{cr}(e) = (e_M - e_m)^{-1} = Z^{-1}, \quad (30)$$

$$\bar{h}_{r(cr)} \equiv \langle h_r(e) \rangle_{cr} = (e_M - e_m)^{-1} \cdot \int_{e_m}^{e_M} h_r(e)\, de, \quad (31)$$

where the subscript "cr" here indicates the critical state for the bulk solid. This state intuitively corresponds to the loose random packing state of bulk solids. This is when a bulk solid is poured into a container, i.e., a silo or a bin, quite randomly. Thus, all possible bulk 'Voronoi Cells' have equal chance of formation under the existing overburden pressure, gravitational effects and interparticle frictional effects. Statistically speaking, for a large number of bulk cells the aggregate ends up with equal number of all possible bulk 'Voronoi Cells' whose void ratio lies in the interval $[e_m, e_M]$ and thus giving rise to a uniform distribution of void ratio in the bulk solids aggregate. Note that the critical void ratio corresponding to this case may be obtained as

$$\bar{e}_{cr} = \int_{e_m}^{e_M} p_{cr}\, e\, de = \frac{1}{2}(e_m + e_M). \quad (32)$$

Therefore for any stored bulk solid an estimate on the gross bulk density $\bar{\varrho}$ may be obtained by the following relation

$$\bar{\varrho} = \left[\frac{2\varrho_s}{2 + e_m + e_M} \right], \quad (33)$$

where ϱ_s is the solid density or the grain density, and e_m and e_M are the void ratios corresponding to the densest and the loosest bulk 'Voronoi Cells'.

4. Vibratory Densification of Stored Bulk Solids

As a bulk solid aggregate is produced by random pouring it produces a uniform distribution density at a maximum entropy. However, this state is extremely unstable under dynamic disturbances, i.e., shaking, tamping, vibrations, and general noise. Under such dynamic disturbances the *loose random packing* state will gravitate towards a denser state commonly referred to as a *dense random packing.* Obviously a loose random packing in transition to a dense random packing witnesses the collapse of looser bulk 'Voronoi Cells' to denser ones and thus a new distribution density emerges. Thus, λ_r starts to vary throughout densification. Since entropy $\bar{S}_{max}$ cannot become any larger throughout the process of densification and compaction, therefore, throughout this process

$$\frac{\partial \bar{S}_{max}}{\partial \lambda_r} < 0, \quad (34)$$

as λ_r is varied to produce the new geometrical distributions of void spaces. From equation (22) it is clear that

$$\frac{\partial^2 Z}{\partial \lambda_r^2} > 0, \forall \lambda_r. \quad (35)$$

Thus from equation (27) and the inequalities (34) and (35) we conclude that throughout vibrating densification of bulk solids λ_r's is always greater than zero and thus the probability distribution density $p(e)$ becomes skewed towards the denser bulk 'Voronoi Cells'. Fig. 3 shows the change in distribution due to vibratory densification.

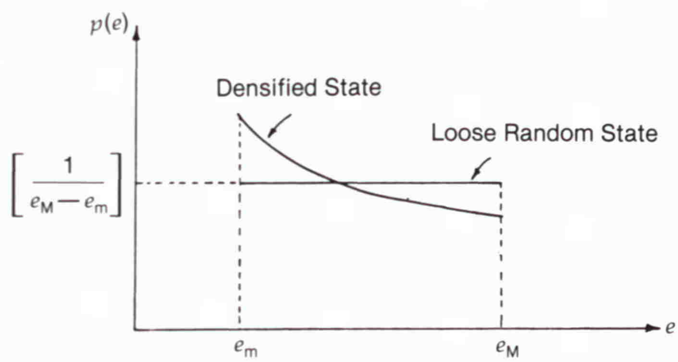


Fig. 3: Change in distribution density $p(e)$ due to vibratory densification

Our conclusions are in agreement with numerically obtained results and conclusions by Finney [3, 4] who states that this skewness is to be expected, the restriction of the proportion of high density cells reflecting the limited availability of space in a packing whose overall density is close to the upper limit, i.e., dense random packing state.

5. Plane Shearing of Bulk Solids and the Critical State

Soil and bulk solids mechanicists have long known the occurrence of a critical state in sufficiently and unidirectionally plane sheared samples of such materials. The reader is referred to Roscoe, Schofield and Wroth [22] and Kezdi [8] for discussion on this concept. As also explained by Shahinpoor [1] and [23] the critical void ratios reached in such critical states tend to remain constant with further shearing in the same direction and have values remarkably close to the values of critical void ratios corresponding to *loose random packings.* It is well known that, unless one starts with a dense bulk solid, in the field of plane shear, bulk solids always undergo some initial densification as reported by Kezdi [8], Roscoe, Schofield and Wroth [22] and Leslie Youd [24]. However, eventually shear induces dilation and the aggregate tends to become looser. Intuitively, it is evident that there are great similarities between random pouring and random shearing of individual bulk 'Voronoi Cells' in bulk solids. Essentially shearing of bulk solids will finally amount to a completely random process of creation and collapse of bulk 'Voronoi Cells' under the existing confining pressure in such a manner that all possible bulk 'Voronoi Cells' will have equal chances of being created or being annihilated or collapsed, and thus there will continuously exist equal numbers of all possible bulk 'Voronoi Cells'. This of course would correspond to a uniform distribution of such cells throughout the bulk solid's volume and a globally maximum entropy as discussed before. Clearly, the initial void ratio will have no effect on the final distribution of void ratios in the shearing process. This fact has recently been demonstrated by Shahinpoor [23] and Kanatani [20] who has presented an ergodic theoretic argument on the growth of entropy and dilation of granular materials with shearing.

6. Conclusions

The concept of bulk 'Voronoi Cells' can be successfully employed for the mechanical description and handling of bulk solids. The critical void ratios obtained in simple shearing of bulk solids should be the same as the critical void ratios obtained for the *loose random packing* of bulk solids under the same confining pressure. The distribution density for such critical states is uniform. However, vibratory and noise densification of bulk solids causes such uniform distributions to become Maxwellian and skewed towards the population of denser bulk 'Voronoi Cells'. An equation for the determination of critical bulk densities was given and an inequality governing the process of vibratory densification of bulk solids was derived. Some typical bulk 'Voronoi Cells' for both two-dimensional and three-dimensional bulk assemblies were also presented.

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